On the Evaluation of Scalarproducts of Nonlinear Spinorfield State Functionals

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The metrical structure of the linear state space of a quantized nonlinear field cannot be given a priori. Rather it is determined by the dynamics of the field itself. For the evaluation of state norms and scalarproducts this metric must be known. In functional quantum theory the metrical structure is expressed by the metric tensor $\mathfrak{G}(j)$ in functional space. Equivalent to the knowledge of $\mathfrak{G}(j)$ is the knowledge of the set of dual state functionals $\{|\mathfrak{S}(j,a)\rangle\}$ together with the corresponding original state functionals $\{|\mathfrak{F}(j,a)\rangle\}$. In preceding papers attempts were made to calculate $\mathfrak{G}(j)$. In this paper an approach is made to determine the dual state functionals directly. Equations are derived which have to be satisfied by the dual functionals. The method works in those state sectors which are characterized by real (monopole) particles or monopole ghosts, while it does not work for multipole ghost states. Norm calculations are performed for local monopole fermion states and local monopole boson states of the lepton-quark model derived in a preceding paper.

The mathematical and physical structures of quantum theory are based on the concept of linear spaces which are equipped with suitable inner products, i.e. scalar products of the corresponding state vectors. Due to the necessity of a probabilistic interpretation and for the evaluation of numerical results with respect to explicit state representations, the scalar products play a fundamental role in quantum theory. However, due to the lack of suitable explicit state representations beyond the free particle states, in relativistic quantum field theory an attempt was made in the past decades to obtain general and numerical information about quantum observables without the use of scalarproducts with respect to explicit state representations, i.e. to obtain representation-free results. The recent developments of relativistic quantum field theory show that such a program cannot be performed consistently. This stems essentially from two reasons:

i) the occurrence of indefinite metric;

ii) the occurrence of bound states.

We first discuss ii). If bound states occur in a relativistic quantum field theory, i.e. if spatially extended particles occur, explicit state representations must be used. This is demonstrated by the work of Huang and Weldon [1] who tried to generalize the L.S.Z.-technique for the derivation of the S-matrix to include extended composite particles. The L.S.Z.-technique is one of the attempts to work representation-free. But in this case explicit state representations of the composite particles are needed in order to obtain the corresponding S-matrix elements by projection by means of scalarproducts. It will be shown later that Huang and Weldon's assumptions are not correct and their conclusions are partly wrong. Nevertheless it remains the fact that a representation-free calculation of the S-matrix for composite particles is not possible. Concerning i), the indefinite metric is used for an invariant formulation of gauge theories or for the regularization of renormalizable and nonrenormalizable quantum field theories. In both cases explicit representations of the state spaces are needed in order to separate the physical parts from the unphysical parts of the state space and to maintain the probabilistic interpretation. In addition, indefinite metric appears at unforeseen occasions. For instance the Bethe-Salpeter bound state norms are partly negative definite, cf. Nakanishi [2], which destroys the selfconsistency of the norm derivation for these states.

Summarizing we may conclude that in spite of the success of representation-free techniques in relativistic quantum field theory for a complete description of relativistic quantized nonlinear fields explicit state representations must be constructed.

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In order to solve this problem Stumpf and coworkers, cf. Stumpf [3], developed the functional quantum theory. Functional quantum theory is a new formulation of quantum theory and a new field theoretic calculation method which allows the treatment of quantized fields with positive metric as well as with indefinite metric beyond perturbation theory, i.e. with inclusion of bound states and their interactions. In this approach explicit state representations and their scalarproducts are used from the beginning, i.e. this theory in principle works like ordinary quantum mechanics with the only difference that the calculational scheme is completely different from ordinary quantum mechanics, as in functional quantum theory all calculations must be explicitly relativistically invariant. It is obvious that such a new approach requires a thorough analysis which can only gradually be achieved. Hence a lot of papers have already been concerned with this problem, cf. [3] and Stumpf [4]. Apart from special calculation techniques the general conclusion consists in the fact that any nonlinear quantized field produces its own metrical structure and its own characteristic state space which are determined by the dynamics themselves. Therefore, it is not admissible to postulate for a quantized nonlinear field a certain metrical structure of the state space without regard to the dynamics of the field. Hence states and scalarproducts must be derived by dynamical calculations. In [4] an attempt was made to derive the dynamical structure of the corresponding state space by a perturbation theoretical calculation of the metrical fundamental tensor

$$g_{nm} := \langle 0 \,|\, N\,\psi(x_1) \ldots \psi(x_n)\, N\,\psi(x_1') \ldots \psi(x_m') \,|\, 0 \rangle$$

of a nonlinear spinorfield $\psi(x)$ starting from the one-particle sector of pointlike fermions. If one tries to generalize this formalism to include bound states of the elementary fermions of the spinorfield, then the norms of bound states must be known in order to solve this problem. We will show that this practically leads to the construction of the original state functional as well as of its dual representation and we will give equations which allow the calculation of both types of state functionals provided the extended bound states under consideration are no multipole states with respect to their center of mass momentum. If both functionals are known, the scalarproduct can be established without reference to g_{nm}. In this way we obtain a direct calculation method of the scalarproduct and in addition a complete representation of the spinorfield state functionals. The general method is discussed in Section 1, while in Sections 2 and 3 some applications are given.

The concept of functional quantum theory was taken over by Wahl and coworkers in order to treat nonrelativistic many-particle systems by means of one-time normal ordered functionals. For a class of simple groundstates Wahl and Feist [5] could deduce the explicit form of the metrical fundamental tensor in functional space. But due to the one-time formalism and the additional restrictions their method is not applicable to the relativistic problem treated here.

1. Calculation of Dual Functional States

Let $\{|a\rangle\}$ be a complete set of states of a linear space $\mathscr{H}$ which describes the quantum theory of a relativistic field system Σ under consideration. Then the state functionals $|\mathfrak{F}(j,a)\rangle$ are defined by an isomorphic map of $\mathscr{H}$ into a functional state space $\mathfrak{H}$, cf. [3]. The scalarproduct in $\mathscr{H}$ is assumed to be an invariant under this map. To achieve this invariance the scalarproduct of the map in the functional state space $\mathfrak{H}$ has in general to be equipped with a metrical fundamental tensor $\mathfrak{G}(j)$, where $\mathfrak{G}(j)$ depends on the system under consideration.

$$\langle a \,|\, b \rangle = \langle \mathfrak{F}(j,a) \,|\, \mathfrak{G}(j)^{-1} \,|\, \mathfrak{F}(j,b) \rangle . \tag{1.1}$$

Defining the dual state functional $\langle \mathfrak{S}(j,a)|$ by

$$\langle \mathfrak{S}(j,a)| := \langle \mathfrak{F}(j,a) \,|\, \mathfrak{G}(j)^{-1} \tag{1.2}$$

the scalarproduct (1.1) can be expressed by

$$\begin{aligned} \langle a \,|\, b \rangle &= \langle \mathfrak{S}(j,a) \,|\, \mathfrak{F}(j,b) \rangle \\ &= \langle \mathfrak{S}(j,a) \,|\, \mathfrak{G}(j) \,|\, \mathfrak{S}(j,b) \rangle , \end{aligned} \tag{1.3}$$

and we see that for a complete definition of the map, a state $|a\rangle$ must be characterized by $|\mathfrak{F}(j,a)\rangle$ as well as by its dual state functional $\langle \mathfrak{S}(j,a)|$.

Without referring to functionals, Nishijima [6] first used the notion of co- and contravariant components of a state $|a\rangle$, at which the contravariant components of $|a\rangle$ were taken for a common time t. The necessary generalization to a many-time formalism and a complete formulation of the concept of functional quantum theory, written in components, were first given by Stumpf [7] and subsequently formulated in functional space, cf. [3]. By

applying and developing the Green function formalism and Bethe-Salpeter amplitude normalization Nishijima [8] tried to avoid the contravariant state components completely. But as has already been pointed out, due to the results of Nakanishi, the normalization of Bethe-Salpeter amplitudes is inconsistent for positive metric field theories and cannot be performed for indefinite metric theories. In addition, even if this procedure is consistent, it is not sufficient for the needs of functional quantum theory, since the normalization and the scalarproducts of complete state functionals and not only of parts of them are required. Hence a more thorough analysis of the problem must be given. In [4] an approach was made to calculate the metrical fundamental tensor $\mathfrak{G}(j)$ for the field under consideration. In this paper we attempt to calculate the dual functionals $\langle \mathfrak{S}(j,a)|$ directly. For this approach the distinction between symmetric and unsymmetric representations of selfadjoint operators is essential. This distinction was first fully recognized by Maison and Stumpf [9] for the case of a one-time description of the anharmonic oscillator by means of state functionals, resp. generating functions, where a proof of the convergence for the symmetric N.T.D.-procedure was given. We shall show that by taking into account this distinction a complete determination of the co- as well as of the contravariant components is possible, provided that we work with selfadjoint operators. The selfadjointness of the corresponding operators in a relativistic quantum field theory with indefinite metric cannot be assumed a priori. Rather it has to be investigated in each particle sector of the general state space which is separated from the other sectors by a superselection rule. It is then a hypothesis of nonlinear spinorfield quantum theory with radical unification and confinement, that the indefinite metric occurs in a dangerous way only in the elementary unobservable one-particle local fermion sector, while the bound states of these local confined fermions which are observable states show a completely regular behavior. The investigation of this behavior can be done by the study of the solutions of the $|\mathfrak{F}(j,a)\rangle$ functional equation. But a general proof of this hypothesis has not yet been given. Therefore, we demonstrate the determination of the dual functionals for the case of a completely continuous selfadjoint operator and generalize the results obtained in this way to the nonlinear spinorfield functionals without proof.

Let K be a completely continuous selfadjoint operator and let $\{|a_i\rangle\}$ be a set of eigenvectors of K with $K|a_i\rangle = \omega_i|a_i\rangle$, $1 \leqq i \leqq n$, then this set can be completed to form a basis of a corresponding Hilbert space $\mathscr{A}$ and the following relations hold

$$K = \sum_{i=1}^{n} |a_i\rangle\, \omega_i \langle a_i|\,, \tag{1.4}$$

$$\mathbf{1} = \sum_{i=1}^{m} |a_i\rangle\langle a_i|\,, \tag{1.5}$$

$$\langle a_i|a_j\rangle = \delta_{ij}; \quad 1 \leqq i,j \leqq m\,, \quad m \geqq n \tag{1.6}$$

cf. Großmann [10]. For simplicity we assume $m = n$ and study an unsymmetrical representation of K by expanding the eigenvectors $|a_i\rangle$ into a series with respect to a nonorthonormal but complete basis $\{|\xi_\alpha\rangle, 1 \leqq \alpha \leqq n\}$ of the Hilbert space $\mathscr{A}$. Then a dual basis $\{|\xi^\beta\rangle, 1 \leqq \beta \leqq n\}$ can be introduced which follows from the conditions $\langle \xi_\alpha|\xi^\beta\rangle = \delta_\alpha^\beta$, $1 \leqq \alpha, \beta \leqq n$ and the state $|a_i\rangle$ can equivalently be represented by

$$|a_i\rangle = \sum_\alpha \varphi_i(\alpha)\,|\xi^\alpha\rangle = \sum_\beta \sigma_i(\beta)\,|\xi_\beta\rangle\,. \tag{1.7}$$

Multiplication of (1.7) from the left by $\langle \xi^\gamma|$ gives

$$\varphi_i(\gamma) = \sum_\beta \langle \xi_\gamma|\xi_\beta\rangle\, \sigma_i(\beta)\,. \tag{1.8}$$

Furthermore, the combination of (1.6) and (1.7) yields

$$\sum_\alpha \varphi_i(\alpha)^\times \sigma_j(\alpha) = \delta_{ij}\,. \tag{1.9}$$

Equations (1.8) and (1.9) are those equations which have so far been studied in functional quantum theory. We now derive further equations for the co- and contravariant components. We substitute into the eigenvalue equation

$$\sum_l |a_l\rangle\, \omega_l \langle a_l|a_i\rangle = \omega_i\,|a_i\rangle \tag{1.10}$$

the expansions (1.7) and obtain

$$\left\{\sum_l \left[\sum_\alpha \varphi_l(\alpha)\,|\xi^\alpha\rangle\, \omega_l \sum_\beta \sigma_l(\beta)^\times \langle \xi_\beta|\right]\right\} \sum_\gamma \varphi_i(\gamma)\,|\xi^\gamma\rangle = \omega_i \sum_\gamma \varphi_i(\gamma)\,|\xi^\gamma\rangle \tag{1.11}$$

whereas the adjoint equation of (1.10) leads to

$$\sum_\gamma \sigma_i(\gamma)^\times \langle \xi_\gamma| \left\{\sum_l \left[\sum_\alpha \varphi_l(\alpha)\,|\xi^\alpha\rangle\, \omega_l \sum_\beta \sigma_l(\beta)^\times \langle \xi_\beta|\right]\right\} = \omega_i \sum_\gamma \sigma_i(\gamma)^\times \langle \xi_\gamma|\,. \tag{1.12}$$

Multiplication of (1.11) from the left with $\langle \xi_\varrho |$ and of (1.12) from the right with $|\xi^\varrho\rangle$ then gives

$$\sum_\beta \left[\sum_l \varphi_l(\varrho)\, \omega_l\, \sigma_l(\beta)^\times \right] \varphi_i(\beta) = \omega_i\, \varphi_i(\varrho) \tag{1.13}$$

resp.

$$\sum_\alpha \sigma_i(\alpha)^\times \left[\sum_l \varphi_l(\alpha)\, \omega_l\, \sigma_l(\varrho)^\times \right] = \omega_i\, \sigma_i(\varrho)^\times \tag{1.14}$$

i.e. the co-resp. contravariant components of $|a_i\rangle$ are the right-hand resp. left-hand solutions of the eigenvalue equation for an unsymmetrical representation of K.

We now transfer this result to quantum field theory. There we have, cf. [3, 4]

$$\begin{aligned} |a_i\rangle &= \sum_\alpha \frac{1}{\alpha!} \int \sigma_\alpha(x_1 \dots x_\alpha | a_i)\, | x_1 \dots x_\alpha \rangle_\alpha \, \mathrm{d}^4x_1 \dots \mathrm{d}^4x_\alpha \triangleq \sum_\alpha \sigma_i(\alpha) |\xi_\alpha\rangle \\ &= \sum_\beta \frac{1}{\beta!} \int \varphi_\beta(x_1 \dots x_\beta | a_i)\, | x_1 \dots x_\beta \rangle^\beta \, \mathrm{d}^4x_1 \dots \mathrm{d}^4x_\beta \triangleq \sum_\beta \varphi_i(\beta) |\xi^\beta\rangle \end{aligned} \tag{1.15}$$

with

$$\begin{aligned} \sigma_\alpha(x_1 \dots x_\alpha | a_i) &:= {}^\alpha\langle x_1 \dots x_\alpha | a_i\rangle \triangleq \langle \xi^\alpha | a_i\rangle , \\ \varphi_\beta(x_1 \dots x_\beta | a_i) &:= {}_\beta\langle x_1 \dots x_\beta | a_i\rangle \triangleq \langle \xi_\beta | a_i\rangle , \end{aligned} \tag{1.16}$$

and

$${}_\beta\langle x_1 \dots x_\beta | := \langle 0 | N \psi(x_1) \dots \psi(x_\beta) \triangleq \langle \xi_\beta | , \quad 1 \leqq \beta < \infty , \tag{1.17}$$

where the base set (1.17) is nonorthogonal for a nonlinear field. We consider a selfadjoint operator $\boldsymbol{K}$ of an observable of the quantum field under consideration and assume that the set $\{|a_i\rangle\}$ represents the set of eigenstates of $\boldsymbol{K}$, i.e.

$$\boldsymbol{K} |a_i\rangle = k_i |a_i\rangle . \tag{1.18}$$

If the covariant representation of $|a_i\rangle$ is substituted into (1.18), then the system for the calculation of the coefficients $\{\varphi_i(\beta), 1 \leqq \beta < \infty\}$ is obtained in the following way: The operator $\boldsymbol{K}$ is applied to the single φ-functions and the results are collected with respect to $|x_1 \dots x_\beta\rangle^\beta$; i.e. we obtain in a symbolic notation

$$\boldsymbol{K} |a_i\rangle = \sum_\beta \frac{1}{\beta!} \left[\sum_\varkappa (\boldsymbol{K})_{\beta, \beta+\varkappa} \varphi_i(x | \beta + \varkappa) \right] |x\rangle^\beta = k_i \sum_\beta \frac{1}{\beta!} \varphi_i(x | \beta) |x\rangle^\beta . \tag{1.19}$$

Multiplication of (1.19) from the left with ${}_\alpha\langle x|$ then yields

$$\sum_\varkappa (\boldsymbol{K})_{\alpha, \alpha+\varkappa} \varphi_i(x | \alpha + \varkappa) = k_i\, \varphi_i(x | \alpha) . \tag{1.20}$$

Obviously this is the analogous procedure which led to the derivation of (1.13). From this we conclude that the corresponding σ-functions must be the lefthand solutions of (1.20). This can be expressed in a compact version by the use of state functionals. When doing so, it must be emphasized that the orthonormalized basis of the state functionals in the functional space $\mathfrak{H}$ is only a mathematical tool and has nothing to do with the nonorthogonality of the basis in $\mathscr{H}$. This difference of the properties of the base vectors in $\mathscr{H}$ resp. $\mathfrak{H}$ is reflected by the fact that the physical scalarproduct in $\mathfrak{H}$ has to be defined with respect to $\mathfrak{G}(j)$, but it does not play a role for the derivation of dynamical equations, as there the dual base sets are used which are always orthonormalized.

According to [3] the functional equation for the nonlinear spinorfield $\mathfrak{F}$-state functional reads

$$[D(x)\, \partial(x) + V\, \mathrm{d}(x)\, \mathrm{d}(x)\, \mathrm{d}(x)] |\mathfrak{F}(j, a)\rangle = 0 . \tag{1.21}$$

This equation is no eigenvalue equation for an observable. Starting with the many-time functionals for the anharmonic oscillator in [9] a functional eigenvalue equation for the energy was derived. The infinitesimal generators of the Poincaré group in functional space were formulated by Stumpf, Scheerer and

Märtl [11], and Stumpf [12] derived a functional eigenvalue equation for the mass. We first apply the Green function $D^{-1}(x)$ to (1.21) and iterate equation (1.21) once. This gives

$$\begin{aligned}\partial(x)\,\partial(y)\,|\,\mathfrak{F}(j,a)\rangle = &-[\partial(y), \int G(x-x')\,V\,\mathrm{d}(x')\,\mathrm{d}(x')\,\mathrm{d}(x')\,\mathrm{d}^4x']_+\,|\,\mathfrak{F}(j\,a)\rangle \\ &-[\int G(y-y')\,V\,\mathrm{d}(y')\,\mathrm{d}(y')\,\mathrm{d}(y')\,\mathrm{d}^4y' \int G(x-x')\,V\,\mathrm{d}(x')\,\mathrm{d}(x')\,\mathrm{d}(x')\,\mathrm{d}^4x']\,|\,\mathfrak{F}(j,a)\rangle\,.\end{aligned} \tag{1.22}$$

We now multiply (1.22) by $-j(x)j(y)\,\partial_{x\mu}\partial_y^\mu$ and integrate over x ynd y. In a symbolic notation this yields

$$\mathfrak{P}^2(j,\partial)\,|\,\mathfrak{F}(j,a)\rangle = \boldsymbol{P}^2(j,\partial)\,|\,\mathfrak{F}(j,a)\rangle \tag{1.23}$$

with

$$\begin{aligned}\boldsymbol{P}^2(j,\partial) := &\int j(x)\,j(y)\,\partial_{x\mu}\,\partial_y^\mu\,[\partial(y), G(x-x')\,V\,\mathrm{d}(x')\,\mathrm{d}(x')\,\mathrm{d}(x')]_+\,\mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4x' \\ &+\int j(x)\,j(y)\,\partial_{x\mu}\,\partial_y^\mu\,[G(y-y')\,V\,\mathrm{d}(y')\,\mathrm{d}(y')\,\mathrm{d}(y')\,G(x-x')\,V\,\mathrm{d}(x')\,\mathrm{d}(x')\,\mathrm{d}(x')]\,\mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4x'\,\mathrm{d}^4y\,.\end{aligned} \tag{1.24}$$

As the application of $\mathfrak{P}^2$ on $|\,\mathfrak{F}(j,a)\rangle$ produces the eigenvalue of the total mass square, (1.23) goes over into

$$m^2\,|\,\mathfrak{F}(j,a)\rangle = \boldsymbol{P}^2(j,\partial)\,|\,\mathfrak{F}(j,a)\rangle\,. \tag{1.25}$$

If the operator $\boldsymbol{P}^2$ is assumed to be selfadjoint, the dual functional $|\,\mathfrak{S}(j,a)\rangle$ must be the left-hand solution of this equation, i.e.

$$\langle\mathfrak{S}(j,a)\,|\,m^2 = \langle\mathfrak{S}(j,a)\,|\,\boldsymbol{P}^2(j,\partial)\,. \tag{1.26}$$

This equation is the needed equation for the determination of the dual functional $|\,\mathfrak{S}(j,a)\rangle$ and we will discuss some simple examples in the following sections. It is interesting to note that the $\mathfrak{F}$-functional must in any case satisfy the original equation (1.21) at which as a consequence also the eigenvalue equation (1.23) is satisfied, whereas the $\mathfrak{S}$-functional has to satisfy only the eigenvalue equation (1.26). In spite of this apparent asymmetry we are not free to choose the lowest component of $|\,\mathfrak{S}\rangle$ only orthogonal to the lowest component of $|\,\mathfrak{F}\rangle$ and to put all other components equal to zero, as Huang and Weldon proposed. Rather the special solutions of equation (1.26) express the metrical structure of the field and replace the knowledge of the fundamental tensor $\mathfrak{G}(j)$.

2. Local Fermion State Normalization

To demonstrate the method for the calculation of dual functional states we consider local fermion and local boson states which occur as basic constituents of the quantum states of a nonlinear spinorfield as a lepton-hadron model with quark confinement developed by Stumpf [13]. In this section we treat the local fermions. In the lowest approximation without selfenergy corrections produced by the nonlinearity of the spinorfield, we obtain from (1.21)

$$D(x)\,\partial(x)\,|\,\mathfrak{F}(j,a)\rangle = 0\,. \tag{2.1}$$

For the model under consideration we have

$$D(x) := (-i\,\partial^\varrho\gamma_\varrho + \mu)^2\,(-i\,\partial^\varkappa\gamma_\varkappa + m) \tag{2.2}$$

and by applying the projection into the lepton resp. the quark sector, cf. [13], we get from (2.1)

$$(-i\,\partial^\varkappa\gamma_\varkappa + m)\,\partial_l(x)\,|\,\mathfrak{F}(j_l,l\rangle = 0 \tag{2.3}$$

resp.

$$(-i\,\partial^\varrho\gamma_\varrho + \mu)^2\,\partial_q(x)\,|\,\mathfrak{F}(j_q,q)\rangle = 0\,. \tag{2.4}$$

These equations are solved by

$$\begin{aligned}&|\,\mathfrak{F}(j,a)\rangle = \int \varphi_1(x\,|\,a)\,j_a(x)\,\mathrm{d}^4x\,, \\ &a = l,q\,,\end{aligned} \tag{2.5}$$

where φ_1-functions are solutions of the equations

$$(-i\,\partial^\varkappa\gamma_\varkappa + m)\,\varphi_1(x\,|\,l) = 0 \tag{2.6}$$

resp.

$$(-i\,\partial^\varrho\gamma_\varrho + \mu)^2\,\varphi_1(x\,|\,q) = 0\,. \tag{2.7}$$

Instead of deriving an eigenvalue equation for $\mathfrak{P}^2$ we can equally well look for the eigenvalues of $\mathfrak{P}_0$. By multiplication with $j_l(x)\gamma^0$ and integration over x equation (2.3) goes over into

$$\begin{aligned}&\int j_l(x)\,i\,\frac{\partial}{\partial t}\,\partial_l(x)\,\mathrm{d}^4x\,|\,\mathfrak{F}(j_l,l)\rangle = E\,|\,\mathfrak{F}(j_l,l)\rangle \\ &= \int j_l(x)\,\gamma^0(-i\,\boldsymbol{\gamma}\cdot\nabla + m)\,\partial_l(x)\,\mathrm{d}^4x\,|\,\mathfrak{F}(j_l,l)\rangle\,.\end{aligned} \tag{2.8}$$

Hence the eigenvalue equation for the dual functional reads

$$\begin{aligned}&\langle\mathfrak{S}(j_l,l)\,|\,E \\ &= \langle\mathfrak{S}(j_l,l)\,|\,\int j_l(x)\,\gamma^0\,(-i\,\boldsymbol{\gamma}\cdot\nabla + m)\,\partial_l(x)\,\mathrm{d}^4x\end{aligned} \tag{2.9}$$

and with the ansatz

$$\langle \mathfrak{S}(j_l, l)| = \int \sigma_1(x|l)^{\times}{}_{\mathfrak{F}}\langle 0|\,\partial_l(x)\,\mathrm{d}^4x \tag{2.10}$$

from (2.9) the equation

$$E\,\sigma_1(x|l)^{\times} = \sigma_1(x|l)^{\times}\gamma^0(-i\boldsymbol{\gamma}\cdot\nabla + m) \tag{2.11}$$

follows. To solve this equation we use the ansatz

$$\sigma_1(x|l) = e^{i\mathfrak{k}\cdot\mathfrak{x}} f(t)\,\sigma(\mathfrak{k})\,\gamma^0. \tag{2.12}$$

Then $\sigma(\mathfrak{k})$ must satisfy the equation

$$\sigma(\mathfrak{k})^{\times}[\gamma^0 E - \boldsymbol{\gamma}\cdot\mathfrak{k} - m] = 0 \tag{2.13}$$

while $f(t)$ remains completely arbitrary. But this is just the general solution for the normalization of Dirac spinors, cf. [14]. Assuming $f(t) = \delta(t)$ we obtain the ordinary normalization condition on the hyperplane $x_0 = 0$.

Concerning (2.7) this equation is the spinorial version of the corresponding scalar model of Froissart [15], which has thoroughly been investigated. As the treatment of the spinorial model runs along the same pattern we discuss the scalar model. Its equation is given by

$$(\Box - \mu^2)^2\,\psi(x) = 0 \tag{2.14}$$

and can equivalently be expressed by the system of equations

$$\begin{aligned}(\Box - \mu^2)\,A(x) &= 0\,,\\ (\Box - \mu^2)\,B(x) &= 2\,\lambda^2 A(x)\end{aligned} \tag{2.15}$$

with $B(x) \equiv \psi(x)$. The solution of (2.15) can be represented in terms of the solutions $A_0(x)$ and $B_0(x)$ of the corresponding homogeneous equations. They are given by, cf. Nagy [16],

$$\begin{aligned}A(x) &= A_0(x) \qquad\qquad (2.16)\\ &= \sum_{\mathfrak{k}} (2\,\omega_{\mathfrak{k}})^{-1/2}\,[a(\mathfrak{k})\exp(ikx) + a^{\times}(\mathfrak{k})\exp(-ikx)]\end{aligned}$$

and

$$B(x) = B_0(x) + \frac{\lambda^2}{\mu^2}(2 + x_\mu\,\partial^\mu)\,A_0(x) \tag{2.17}$$

with

$$\begin{aligned}&B_0(x) \qquad\qquad (2.18)\\ &= \sum_{\mathfrak{k}} (2\,\omega_{\mathfrak{k}})^{-1/2}\,[b(\mathfrak{k})\exp(ikx) + b^{\times}(\mathfrak{k})\exp(-ikx)]\,.\end{aligned}$$

The Hamiltonian reads [15, 16]

$$\begin{aligned}H = \sum_{\mathfrak{k}} \omega_{\mathfrak{k}}\Big[&a^{\times}(\mathfrak{k})\,b(\mathfrak{k}) + b^{\times}(\mathfrak{k})\,a(\mathfrak{k}) \qquad (2.19)\\ &+ \frac{\lambda^2}{\mu^2}\,a^{\times}(\mathfrak{k})\,a(\mathfrak{k})\Big] = \sum_{\mathfrak{k}} H(\mathfrak{k})\end{aligned}$$

and canonical quantization yields

$$[a(\mathfrak{k}), b^{+}(\mathfrak{k}')]_{-} = \delta_{\mathfrak{k},\mathfrak{k}'}\,. \tag{2.20}$$

The vacuum state is defined by $H|0\rangle = 0$ and from this it follows that

$$a(\mathfrak{k})|0\rangle = b(\mathfrak{k})|0\rangle = 0 \tag{2.21}$$

must be satisfied as usual. The states

$$|n_{\mathfrak{k}}, m_{\mathfrak{k}}\rangle := [n_{\mathfrak{k}}!\,m_{\mathfrak{k}}!]^{-1/2}\,[a^{+}(\mathfrak{k})]^{n_{\mathfrak{k}}}[b^{+}(\mathfrak{k})]^{m_{\mathfrak{k}}}|0\rangle \tag{2.22}$$

are not orthogonal. Thus in this representation the selfadjointness of H cannot be investigated. If we introduce

$$\begin{aligned}A(\mathfrak{k}) &= (2)^{-1/2}\,[a(\mathfrak{k}) + b(\mathfrak{k})];\\ B(\mathfrak{k}) &= (2)^{-1/2}\,[a(\mathfrak{k}) - b(\mathfrak{k})]\end{aligned} \tag{2.23}$$

the corresponding base system

$$|N_{\mathfrak{k}}, M_{\mathfrak{k}}\rangle := [N_{\mathfrak{k}}!\,M_{\mathfrak{k}}!]^{-1/2}\,[A^{+}(\mathfrak{k})]^{N_{\mathfrak{k}}}[B^{+}(\mathfrak{k})]^{M_{\mathfrak{k}}}|0\rangle \tag{2.24}$$

with $A(\mathfrak{k})|0\rangle = B(\mathfrak{k})|0\rangle = 0$ is orthogonal [16] and for the Hamiltonian we obtain [16]

$$\begin{aligned}H(\mathfrak{k})|N_{\mathfrak{k}}, M_{\mathfrak{k}}\rangle = &\left[(N_{\mathfrak{k}} + M_{\mathfrak{k}}) + \frac{\lambda^2}{2}(N_{\mathfrak{k}} - M_{\mathfrak{k}})\right]\\ &\cdot \omega_{\mathfrak{k}}|N_{\mathfrak{k}}, M_{\mathfrak{k}}\rangle\\ &+ [N_{\mathfrak{k}}(M_{\mathfrak{k}} - 1)]^{1/2}|N_{\mathfrak{k}} - 1, M_{\mathfrak{k}} + 1\rangle\\ &- [M_{\mathfrak{k}}(N_{\mathfrak{k}} + 1)]^{1/2}|N_{\mathfrak{k}} + 1, M_{\mathfrak{k}} - 1\rangle\,. \qquad (2.25)\end{aligned}$$

But in spite of an orthonormal base system the representation of H is unsymmetric and hence H is not selfadjoint. This holds likewise for the spinorial case. As (2.14) leads to two ghost states for any wave vector k, this system is said to describe a dipole ghost. Hence the calculation method for the dual functional which rests on selfadjoint operators cannot be applied to the case of a dipole ghost, resp. a fortiori not to the case of multipole ghost fields. If such situations occur we must return to the calculation of $\mathfrak{S}(j)$. It should, however, be emphasized that monopole ghosts are within the scope of the calculation method for $|\mathfrak{S}(j)\rangle$, as in this case H remains selfadjoint and only the norm becomes negative. The simplest example is the Gupta-Bleuler formalism, but of course in a composite particle theory we are mainly interested in the norm calculation of extended composite particles. The decision whether the $|\mathfrak{S}(j)\rangle$ state can be directly calculated or not follows from an investigation of the eigenvalue equation for the $|\mathfrak{F}(j)\rangle$ functional.

3. Local Boson State Normalization

The lepton-quark model contains in the two-fermion sector bound states of two fermion field operators which are partly identified with the bosons of the strong interactions and partly with the bosons of the electroweak interactions. The latter ones are the simplest boson states which occur at all as they describe local bosons. For these bosons the eigenvalue equation can be solved rigorously and it is of interest to study the dual functionals for this case. As the numerical evaluation has not yet been performed completely, we assume that all eigenvalues have monopole character, i.e. that no multipoles occur in order to secure the applicability of our calculation method for the dual functionals. According to [13] formula (1.21), the eigenvalue equation for the local bosons reads

$$|\mathfrak{F}(s_1 s_2)\rangle = -\boldsymbol{W}(s_1 s_2)\int \delta(s_1+s_2-p_1-p_2)\,|\mathfrak{F}(p_1 p_2)\rangle\,\mathrm{d}^4p_1\,\mathrm{d}^4p_2 \tag{3.1}$$

with

$$|\mathfrak{F}(s_1 s_2)\rangle = \partial(s_1)\,\partial(s_2)\,|\mathfrak{F}(j,b)\rangle \tag{3.2}$$

and according to Sect. 1, we derive from this equation an eigenvalue equation for the selfadjoint operator $\mathfrak{P}^2$. In momentum space this operator is given by

$$\mathfrak{P}^2 := \int j(s_1)\,s_1{}^{\mu}\,\partial(s_1)\,\mathrm{d}^4s_1 \int j(s_2)\,s_{2\mu}\,\partial(s_2)\,\mathrm{d}^4s_2 \tag{3.3}$$

and from (3.1) and (3.2) the equation

$$\begin{aligned}\mathfrak{P}^2|\,\mathfrak{F}\rangle &= m^2|\,\mathfrak{F}\rangle \\ &= \int j(s_1)\,j(s_2)\,s_1{}^{\mu}s_{2\mu}\,\boldsymbol{W}(s_1 s_2)\,\delta(s_1+s_2-p_1-p_2)\,\partial(p_1)\,\partial(p_2)\,|\,\mathfrak{F}\rangle\,\mathrm{d}^4s_1\,\mathrm{d}^4s_2\,\mathrm{d}^4p_1\,\mathrm{d}^4p_2\end{aligned} \tag{3.4}$$

follows. The $|\,\mathfrak{F}\rangle$ state functional has simultaneously to satisfy (3.1) as well as (3.4), while according to Sect. 1 the dual state functional $\langle\mathfrak{S}|$ has to satisfy only equation

$$m^2\langle\mathfrak{S}| = \langle\mathfrak{S}|\int j(s_1)\,j(s_2)\,s_1{}^{\mu}s_{2\mu}\,\boldsymbol{W}(s_1 s_2)\,\delta(s_1+s_2-p_1-p_2)\,\partial(p_1)\,\partial(p_2)\,\mathrm{d}^4s_1\,\mathrm{d}^4s_2\,\mathrm{d}^4p_1\,\mathrm{d}^4p_2\,. \tag{3.5}$$

By the ansatz

$$\langle\mathfrak{S}| := \int \sigma(q_1 q_2)\,{}_{\mathfrak{H}}\langle 0|\,\partial(q_1)\,\partial(q_2)\,\mathrm{d}^4q_1\,\mathrm{d}^4q_2 \tag{3.6}$$

resp.

$$|\,\mathfrak{F}\rangle := \int \varphi(q_1 q_2)\,j(q_1)\,j(q_2)\,|\,0\rangle_{\mathfrak{H}}\,\mathrm{d}^4q_1\,\mathrm{d}^4q_2 \tag{3.7}$$

(3.1), (3.4) and (3.5) lead to the following equations for $\varphi(q_1 q_2)$, resp. $\sigma(q_1 q_2)$

$$\varphi(q_1 q_2) = \boldsymbol{W}(q_1 q_2)\int \delta(q_1+q_2-p_1-p_2)\,\varphi(p_1 p_2)\,\mathrm{d}^4p_1\,\mathrm{d}^4p_2\,, \tag{3.8}$$

$$m^2\varphi(q_1 q_2) = \sum_{ij} q_i{}^{\mu}q_{j\mu}\,\boldsymbol{W}(q_1 q_2)\int \delta(q_1+q_2-p_1-p_2)\,\varphi(p_1 p_2)\,\mathrm{d}^4p_1\,\mathrm{d}^4p_2\,, \tag{3.9}$$

$$m^2\sigma(q_1 q_2) = \sum_{ij}\int \sigma(s_1 s_2)\,s_i{}^{\mu}s_{j\mu}\,\boldsymbol{W}(s_1 s_2)\,\delta(s_1+s_2-q_1-q_2)\,\mathrm{d}^4s_1\,\mathrm{d}^4s_2\,. \tag{3.10}$$

The exact solution of (3.8) was derived in [17]. To formulate it we introduce center-of-mass coordinates for all momentum variables, i.e.

$$s_1 = \tfrac{1}{2}s_{\mathrm{c}} + s_{\mathrm{r}}\,,\quad s_2 = \tfrac{1}{2}s_{\mathrm{c}} - s_{\mathrm{r}}\,, \tag{3.11}$$

etc. Then $\varphi(q_1 q_2)$ is given by

$$\varphi(q_1 q_2) = \delta(P-q_{\mathrm{c}})\,\boldsymbol{W}(P,q_{\mathrm{r}})\,\boldsymbol{\chi}_0 = \delta(P-q_{\mathrm{c}})\,\boldsymbol{\chi}(P,q_{\mathrm{r}})\,, \tag{3.12}$$

where $\boldsymbol{\chi}_0$ is a spintensor of second rank which is determined by the irreducible representation of the Poincaré group for the boson under consideration, and where $P^2 = m^2$ is the boson mass eigenvalue. Due to its derivation (3.9) must also be satisfied by (3.12). Concerning (3.10) we make the ansatz

$$\sigma(q_1 q_2) = \delta(\mathfrak{P}-\mathfrak{q}_{\mathrm{c}})\,f(q_{\mathrm{c}}{}^0)\,\boldsymbol{\Lambda}(P,q_{\mathrm{r}})\,. \tag{3.13}$$

Substitution of (3.13) into (3.10) yields

$$\begin{aligned}&m^2\,\delta(\mathfrak{P}-\mathfrak{q}_{\mathrm{c}})\,f(q_{\mathrm{c}}{}^0)\,\boldsymbol{\Lambda}(P,q_{\mathrm{r}}) \\ &= \int \delta(\mathfrak{P}-\mathfrak{s}_{\mathrm{c}})\,f(s_{\mathrm{c}}{}^0)\,\boldsymbol{\Lambda}(P,s_{\mathrm{r}})\,s_{\mathrm{c}\mu}s_{\mathrm{c}}{}^{\mu}\,\boldsymbol{W}(s_{\mathrm{c}},s_{\mathrm{r}})\,\delta(s_{\mathrm{c}}-q_{\mathrm{c}})\,\mathrm{d}^4s_{\mathrm{c}}\,\mathrm{d}^4s_{\mathrm{r}}\,.\end{aligned} \tag{3.14}$$

Integrating over s_c, we can eliminate the δ-distribution and $f(q_c{}^0)$ and obtain

$$m^2 \mathbf{\Lambda}(P, q_r) = \int (\mathfrak{P} \cdot \mathfrak{P} + q_{c0} q_c{}^0) \mathbf{\Lambda}(P, s_r) \boldsymbol{W}(\mathfrak{P}, q_c{}^0, s_r) \, \mathrm{d}^4 s_r .$$

From the eigenvalue equation (3.9) it follows that due to (3.12) the equation

$$m^2 = \int P^2 W_\varkappa(P, s_r) \, \mathrm{d}^4 s_r \tag{3.16}$$

must be satisfied where $W_\varkappa(P, s_r)$ is the irreducible part of the kernel $\boldsymbol{W}(P, s_r)$ with respect to the boson representation $\boldsymbol{\chi}$. As the solution (3.13) must produce the same eigenvalue, we conclude that (3.15) must have the solution $q_c{}^0 = p^0$ and $\mathbf{\Lambda}(q_r) = \mathbf{\Lambda}_0$. Then (3.13) takes the form

$$\sigma(q_1 q_2) = \delta(\mathfrak{P} - \mathfrak{q}_c) f(q_c{}^0) \mathbf{\Lambda}_0 , \tag{3.17}$$

where $q_c{}^0$ in (3.17) can take arbitrary values, as $f(q_c{}^0)$ is eliminated before the value of $q_c{}^0$ is fixed by the eigenvalue condition (3.16).

The scalarproduct of two boson states with momentum $\mathfrak{P}$ resp. $\mathfrak{P}'$ and the representation indices $\varkappa$, ν resp. $\varkappa' \nu'$ reads

$$\begin{aligned} \langle b, \mathfrak{P}, \varkappa \nu \,|\, b, \mathfrak{P}', \varkappa' \nu' \rangle &= \int \sigma(q_1 q_2 \,|\, b, \mathfrak{P}, \varkappa \nu) \varphi(q_1 q_2 \,|\, b, \mathfrak{P}', \varkappa' \nu') \, \mathrm{d}^4 q_1 \, \mathrm{d}^4 q_2 \\ &= \int \delta(\mathfrak{P} - \mathfrak{q}_c) f(q_c{}^0) \mathbf{\Lambda}_0 \, \delta(P' - q_c) \boldsymbol{W}(P, q_r) \boldsymbol{\chi}_0 \, \mathrm{d}^4 q_c \, \mathrm{d}^4 q_r \\ &= \delta(\mathfrak{P} - \mathfrak{P}') f(p^0) \int \mathbf{\Lambda}_0 \boldsymbol{W}(P, q_r) \boldsymbol{\chi}_0 \, \mathrm{d}^4 q_r . \end{aligned} \tag{3.18}$$

If we now decompose the kernel into irreducible parts, we obtain

$$\int \boldsymbol{W}(P, q_r) \, \mathrm{d}^4 q_r = \sum_\varkappa \int W_\varkappa(P, q_r) \, \mathrm{d}^4 q_r \, \boldsymbol{W}_\varkappa \tag{3.19}$$

and for the irreducible boson representations $\mathbf{\Lambda}_0 = \mathbf{\Lambda}_\varkappa(\nu)$ resp. $\boldsymbol{\chi}_0 = \boldsymbol{\chi}_{\varkappa'}(\nu')$ we have

$$\mathbf{\Lambda}_\varkappa(\nu) \boldsymbol{W}_{\varkappa''} \boldsymbol{\chi}_{\varkappa'}(\nu') = \delta_{\varkappa' \varkappa''} \delta_{\varkappa'' \varkappa} g_\varkappa^{\nu\nu'} \tag{3.20}$$

i.e. we obtain

$$\langle b, \mathfrak{P}, \varkappa \nu \,|\, b, \mathfrak{P}', \varkappa' \nu' \rangle = \delta_{\varkappa\varkappa'} \delta(\mathfrak{P} - \mathfrak{P}') f(p^0) \int W_\varkappa(P, q_r) \, \mathrm{d}^4 q_r \, g_\varkappa^{\nu\nu'} . \tag{3.21}$$

Simultaneously with (3.16) the eigenvalue equation

$$1 = \int W_\varkappa(P, q_r) \, \mathrm{d}^4 q_r \tag{3.22}$$

must be satisfied. Hence (3.21) goes over into

$$\langle b, \mathfrak{P}, \varkappa \nu \,|\, b, \mathfrak{P}', \varkappa' \nu' \rangle = \delta(\mathfrak{P} - \mathfrak{P}') f(p^0) \, \delta_{\varkappa\varkappa'} \, g_\varkappa^{\nu\nu'} , \tag{3.23}$$

where $g_\varkappa^{\nu\nu'}$ is the metrical fundamental tensor in the space of boson states for the representation $\varkappa$ and $f(p_0)$ can be chosen in order to obtain the conventional boson normalization.

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